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Quantum decay processes and Gamov states

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Abstract

By extending the notion of states to functionals acting on the space of observables we obtain a well-defined complex spectral decomposition for the time evolution of quantum-decaying systems, where Gamov states play a fundamental role. It is shown that Gamov vectors are well-defined state functionals and, therefore, they stand on the same footing as plane waves.

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1. Introduction

The study of decaying systems was decisive for establishing quantum mechanics as the theory of microscopical processes. Indeed, it was the seminal paper of Gamov [1] which showed for the first time that quantum mechanics could properly describe the classically forbidden penetrability of a particle through a barrier. Soon afterwards Gamov vectors were introduced as outgoing 'eigenvectors' of the Schrödinger equation. These were related to resonances of the system. The corresponding complex 'eigenvalues' z_0 provided the parameters that determine the resonances, i.e. $\mathrm{Re}(z_0)$ is the position and $-2\,\mathrm{Im}(z_0)$ is the width of the resonance [2]. This theory was, and still is, very appealing since it describes resonances within the same framework as bound states [3]. However, Gamov states are entities which do not belong to quantum mechanics since they cannot be normalized and, perhaps even worse, provide complex probabilities. These features marked a downturn in the use of Gamov vectors, particularly in scattering theory [4], although still some work was done in the subject (see, e.g., [5]).

The theory received new impetus when it was shown that one could 'renormalize' Gamov vectors [6] to form a complete set of states in a space which is not the Hilbert space [7, 8]. New techniques and computing facilities made it possible to carry out that renormalization very conveniently [7–11] and since then Gamov vectors have increasingly been used in various fields of microscopic physics [12–14]. One evaluates all resonances as the poles of the S-matrix but gives physical meaning only to very narrow resonances, for which the imaginary parts of the probabilities (including transition probabilities and cross sections) are negligible [14]. But the fact remains that since Gamov vectors do not belong to Hilbert space they cannot be included in the domain of ordinary quantum mechanics. This can be readily shown (in a heuristic way) by noting that if H is a self-adjoint Hamiltonian with an assumed complex eigenvalue z_0, then the corresponding eigenvector $|f_0\rangle$ satisfies

$$H|f_0\rangle = z_0|f_0\rangle. \tag{1}$$

Conjugating this equation one obtains[4]

$$\langle f_0|H = \overline{z}_0\langle f_0| \tag{2}$$

that is

$$(z_0 - \overline{z}_0)\langle f_0|f_0\rangle = 0. \tag{3}$$

Since z_0 is complex $z_0 \neq \overline{z}_0$ and therefore $\langle f_0|f_0\rangle = 0$. Thus the 'norm' of the Gamov vectors is zero, that is they do not belong to the Hilbert space. Still one can define a norm by introducing an associated vector $|\tilde{f}_0\rangle$ such that [15]

$$\langle \tilde{f}_0|H = z_0\langle \tilde{f}_0| \tag{4}$$

to get

$$(z_0 - z_0)\langle \tilde{f}_0|f_0\rangle = 0 \tag{5}$$

and one can thus define the 'norm' of the Gamov vector as

$$\langle \tilde{f}_0|f_0\rangle = 1. \tag{6}$$

This is the normalization proposed in [6, 7, 9–11], which leads to a metric where the Hilbert internal product $\overline{\phi}\phi = |\phi|^2$ must be substituted by the Berggren internal product ϕ^2, i.e. [7]

$$\langle \tilde{f}_0|f_0\rangle = \int \phi^2\, \mathrm{d}x = 1. \tag{7}$$

But this definition of the norm leaves the question open whether the probability density is the standard one, i.e. $|\phi(x)|^2$, or that corresponding to the Berggren metric, i.e. $\phi(x)^2$. In the first case the probability diverges while in the second complex probabilities are obtained.

There have been several attempts to give a physical interpretation to the imaginary part of the probability (see, e.g., [16, 17]), but limitations to such interpretations have always been found [18]. In fact these apparent solutions take the problem out of the realm of quantum mechanics, introducing notions which are alien to this theory. Moreover, none of the solutions presented so far regarding this problem can be considered definitive, as shown recently in [19].

In this paper we will present a formalism to treat Gamov resonances within a rigorous quantum mechanical framework[5] devoid of any regularization procedure. In order to give a general mathematical structure to Gamov vectors, they will first be defined as *functionals over the space of pure states*, which is an approach similar to that used to define plane waves[6]. We will study the problems presented by this formalism. This study will help us to develop a better treatment of these objects in the second step.

[4] We use an overline to denote a complex conjugate operation.

[5] Note that plane waves are also outside the Hilbert space. In this paper, the word 'rigorous' means 'as rigorous as quantum mechanics with plane waves normalized to Dirac's deltas', i.e. as rigorous as scattering theory.

[6] An ordinary plane wave with energy ω can be symbolized as the ket $|\omega\rangle$ or the bra $\langle\omega|$. It is then clear that these objects do not belong to the Hilbert space, since the orthonormality relation $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$ implies $\langle\omega|\omega\rangle = \infty$. Nevertheless if $|\phi\rangle$ is a ket such that it is well behaved at infinity, $\langle\omega|\phi\rangle$ is well defined, and since $\langle\omega|(\alpha|\phi\rangle + \beta|\phi'\rangle) = \alpha\langle\omega|\phi\rangle + \beta\langle\omega|\phi'\rangle$, then $\langle\omega|$ can be considered as a linear functional over the ket space. Gamov states are of the same nature as plane waves.

A technique to deal with problems involving continuous spectra, and which defines generalized states and generalized observables, was introduced in [20]. This technique was used with success to study quantum decaying problems introducing an approach to equilibrium in statistical quantum mechanics [21]. It was also successfully used to study the phenomenon of decoherence [22, 23]. Here we will apply the same technique to solve the problems associated with Gamov vectors.

In section 2 we consider Gamov vectors as functionals acting on pure states and list the problems introduced by this assumption. In section 3 generalized states are defined as functionals acting on operators representing observables. These generalized states will be used to obtain a real spectral decomposition of the time evolution of the system in section 4. In section 5 a complex spectral decomposition is presented. In this last decomposition Gamov vectors appear naturally as terms of the expansion and as eigenvectors of the Liouville–von Neumann superoperator. In section 6 an application will be presented, and the conclusions are given in section 7.

2. Pure states and Gamov vectors

Let us consider a Hamiltonian H_0 corresponding to free particles with continuous spectrum $[0, \infty)$. We represent by $|\omega\rangle$ ($\langle\omega|$) the right (left) generalized eigenvector of H_0 with eigenvalue ω

$$H_0|\omega\rangle = \omega|\omega\rangle \qquad \langle\omega|H_0 = \omega\langle\omega| \qquad 0 \leqslant \omega < \infty. \tag{8}$$

We also assume that the right and left eigenvectors form an orthogonal complete system, as usual [24], i.e.

$$I = \int_0^\infty \mathrm{d}\omega\, |\omega\rangle\langle\omega| \qquad \langle\omega|\omega'\rangle = \delta(\omega - \omega') \tag{9}$$

where I is the identity operator.

The eigenvectors of H_0 form the basis of what we will call the 'H_0 representation' of the quantum system

$$H_0 = \int \mathrm{d}\omega\, \omega|\omega\rangle\langle\omega|. \tag{10}$$

The full Hamiltonian H of the interacting system will be

$$H = H_0 + V = \int \mathrm{d}\omega\, \omega|\omega\rangle\langle\omega| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, V_{\omega\omega'}|\omega\rangle\langle\omega'| \tag{11}$$

where $V_{\omega\omega'} = \langle\omega|V|\omega'\rangle$ is a regular function of the variables ω and ω'. For simplicity we assume that H also has the continuous spectrum $[0, \infty)$.

Since the time evolution of the system is determined by the Hamiltonian H, it is convenient to change to a representation in terms of the eigenvectors of H (the 'H representation'). For each eigenvector $|\omega\rangle$ of the Hamiltonian H_0 there is an eigenvector $|\omega^+\rangle$ of the Hamiltonian H as given by the Lippmann–Schwinger equation, i.e.

$$|\omega^+\rangle = |\omega\rangle + \frac{1}{\omega + \mathrm{i}0 - H} V|\omega\rangle \tag{12}$$

The corresponding 'bra' is given by

$$\langle\omega^+| = \langle\omega| + \langle\omega|V \frac{1}{\omega - \mathrm{i}0 - H} \tag{13}$$

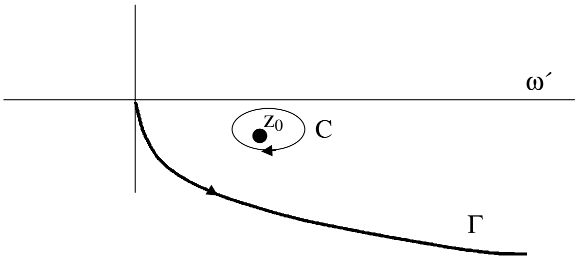

Figure 1. Complex contour on the lower complex energy plane used in our evaluation of integrals. The energy z_0 is the pole assumed to be simple.

We are going to consider a physical system for which the vectors $|\omega^+\rangle$ also generate a complete orthonormal basis

$$\langle\omega^+|\omega'^+\rangle = \delta(\omega-\omega') \qquad H = \int \mathrm{d}\omega\,\omega|\omega^+\rangle\langle\omega^+| \qquad I = \int \mathrm{d}\omega\,|\omega^+\rangle\langle\omega^+|. \tag{14}$$

The probability that a pure state $|\varphi\rangle$ is in the pure state $|\psi\rangle$ at time t is

$$P(t) = |A(t)|^2 \qquad A(t) = \langle\psi|\exp(-\mathrm{i}Ht)|\varphi\rangle = \int_0^\infty \mathrm{d}\omega'\,\exp(-\mathrm{i}\omega' t)\langle\psi|\omega'^+\rangle\langle\omega'^+|\varphi\rangle. \tag{15}$$

Let us assume that the analytic extension to the lower complex half plane of the variable ω' in the integrand of the previous expression has a simple pole at $z = z_0$ in the lower complex half plane, very close to the positive real axis. In this case the integral in equation (15) will have a dominant contribution from the values of ω' close to $\omega_0 \equiv \mathrm{Re}(z_0)$. To describe these resonant effects, it is convenient to deform the domain of integration $[0, +\infty)$ for ω' to a convenient curve in the complex plane[7]. To perform this deformation, we need the analytic extensions $|z\rangle$, $\langle\tilde{z}|$, $|z^+\rangle$ and $\langle\tilde{z}^+|$, of the eigenvectors $|\omega\rangle$, $\langle\omega|$, $|\omega^+\rangle$ and $\langle\omega^+|$. All these objects can be considered only as *functionals* acting over the usual wave vectors. That is, if $\varphi : [0, +\infty) \to \mathbb{C}$ [8] is a wavefunction in the H_0 representation, the 'bra' $\langle\omega|$ is a *linear functional* whose action on φ is defined by[9]

$$\langle\omega|\varphi\rangle \equiv \varphi(\omega). \tag{16}$$

Since our objects will be mainly complex, we have to extend the functionals above to the complex plane. In the domain of the complex plane for which the analytic extension of the function φ is well behaved, we define the linear functional $\langle\tilde{z}|$ through the equation

$$\langle\tilde{z}|\varphi\rangle \equiv \varphi(z) \tag{17}$$

i.e. the functional $\langle\tilde{z}|$ acting on the function $\varphi : [0, +\infty) \to \mathbb{C}$ gives the value of the analytic extension of the function φ at point z of the complex plane.

We are going to choose $\varphi : [0, +\infty) \to \mathbb{C}$ in the class of Schwartz functions having a well-defined analytic extension to the domain D_Γ between $[0, +\infty)$ and the curve Γ in the lower complex half plane (see figure 1).

[7] The contour deformation is equivalent to the method of analytic dilations [11] if the curve is a straight line in the lower complex half plane and the analytic extensions $\varphi(z)$ of the wavefunctions $\varphi(\omega)$ in the 'H_0 representation' go to zero for $|z| \to \infty$.

[8] $\mathbb{C}$ denotes the complex plane.

[9] Through this paper, following the standard tradition in physical literature, the Dirac bra–ket notation is used to denote Hilbert space products such as $\langle\psi|\varphi\rangle \doteq \int_0^\infty \mathrm{d}\omega\,\overline{\psi(\omega)}\varphi(\omega)$, the left action of linear functionals such as $\langle\omega|\varphi\rangle \doteq \varphi(\omega)$, or the right action of antilinear functionals such as $\langle\psi|\omega\rangle \doteq \overline{\psi(\omega)}$. (For details see [30, 31].)

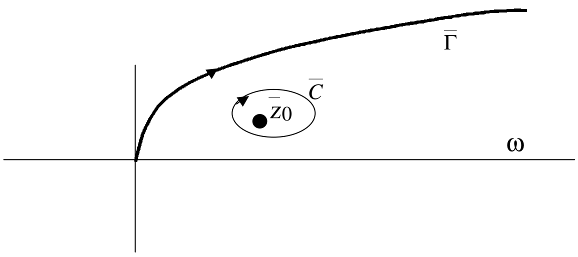

Figure 2. Complex contour on the upper complex energy plane used in our evaluation of integrals. The energy $\overline{z}_0$ is the pole assumed to be simple.

Analogously, the 'ket' $|\omega\rangle$ is an *antilinear functional* defined by

$$\langle\psi|\omega\rangle \equiv \overline{\psi}(\omega) \tag{18}$$

where the function $\overline{\psi} : [0, +\infty) \to \mathbb{C}$ is defined by $\overline{\psi}(\omega) \equiv \overline{\psi(\omega)}$.

In the domain of the complex plane for which the analytic extension of the function $\overline{\psi}$ is well defined, we define the antilinear functional $|z\rangle$ through the relation

$$\langle\psi|z\rangle \equiv \overline{\psi}(z) = \overline{\psi(\overline{z})} \tag{19}$$

i.e. the functional $|z\rangle$, acting on the function $\overline{\psi} : [0, +\infty) \to \mathbb{C}$, gives the value of the analytic extension of the function $\overline{\psi}$ at point z of the complex plane.

We are interested in functions $\overline{\psi} : [0, +\infty) \to \mathbb{C}$ in the class of Schwartz functions having well-defined analytic extensions to the domain D_Γ. Therefore, $\psi : [0, +\infty) \to \mathbb{C}$ should have a well-defined analytic extension in the domain $D_{\overline{\Gamma}}$ between $\mathbb{R}^+$ and the curve $\overline{\Gamma}$ in the upper complex half plane (see figure 2).

In appendix A we prove that the functionals $\langle z|$ and $|\widetilde{z}\rangle$, defined by the usual relations

$$\langle z|\varphi\rangle \equiv \overline{\langle\varphi|z\rangle} \qquad \langle\varphi|\widetilde{z}\,\rangle \equiv \overline{\langle\,\widetilde{z}|\varphi\rangle} \tag{20}$$

verify

$$\langle z| = \langle\,\widetilde{\overline{z}}| \qquad |\widetilde{z}\,\rangle = |\overline{z}\rangle. \tag{21}$$

From these results it follows that $\langle\,\widetilde{z}| = \langle\overline{z}| \neq \langle z|$, i.e. if z is a complex number then $\langle\,\widetilde{z}|$ is not the adjoint of $|z\rangle$. This property justifies the use of a tilde ($\sim$) in the definition given in equation (17), which would not be necessary for real values $z = \omega \in [0, +\infty)$ where $\langle\,\widetilde{\omega}| = \langle\overline{\omega}| = \langle\omega|$.

The *resolvent* $R(z) \equiv (z - H)^{-1}$ is an analytic function[10] of the complex variable z, except for a cut in $[0, +\infty)$. According to equations (12) and (13) we have

$$|\omega^+\rangle = |\omega\rangle + R(\omega + \mathrm{i}0)V|\omega\rangle \qquad \langle\omega^+| = \langle\omega| + \langle\omega|V R(\omega - \mathrm{i}0). \tag{22}$$

Therefore the analytic extensions of $|\omega^+\rangle$ and $\langle\omega^+|$ involve the analytic extensions of the resolvent. We define the analytic extension $R^+(z)$ $(R^-(z))$ of the resolvent $R(z)$ from the upper (lower) to the lower (upper) complex half plane as[11]

$$R^+(z) \equiv \begin{cases} R(z) & z \in \mathbb{C}^+ \doteq \{z \in \mathbb{C}/\,\mathrm{Im}(z) > 0\} \\ \mathrm{cont}_{s\in\mathbb{C}^+\to z} R(s) & z \in \mathbb{C}^- \doteq \{z \in \mathbb{C}/\,\mathrm{Im}(z) < 0\} \end{cases} \tag{23}$$

[10] More precisely: it is analytical in the so-called physical sheet of the corresponding Riemann surface but it has poles in the other sheet, that is in the so-called unphysical sheet.

[11] We use the notation $\mathrm{cont}_{s\in\mathbb{C}^+\to z}$ to indicate the analytic continuation of a function defined in a point s of the upper plane to a point z (which may be in the lower plane). The analytic extension of an operator depending on a complex variable z should always be understood in the weak sense. For example, $\langle\varphi|\mathrm{cont}_{s\in\mathbb{C}^+\to z} R(s)|\psi\rangle \equiv \mathrm{cont}_{s\in\mathbb{C}^+\to z}\langle\varphi|R(s)|\psi\rangle$, where φ and ψ are suitable test functions.

$$R^{-}(z) \equiv \begin{cases} \mathrm{cont}_{s\in\mathbb{C}^{-}\to z} R(s) & z \in \mathbb{C}^{+} \\ R(z) & z \in \mathbb{C}^{-} \end{cases}. \tag{24}$$

From the definitions of $|\omega^{+}\rangle$ and $\langle\omega^{+}|$ given in equations (22) and of the analytic extensions $R^{+}(z)$ and $R^{-}(z)$ of the resolvent given in equations (23) and (24), we can obtain the corresponding analytic extensions of the Lippmann–Schwinger equations

$$|z^{+}\rangle = |z\rangle + R^{+}(z)V|z\rangle \qquad \langle\widetilde{z}^{+}| = \langle\widetilde{z}| + \langle\widetilde{z}|VR^{-}(z). \tag{25}$$

In appendix A we prove that the functionals $\langle z^{+}|$ and $|\widetilde{z}^{+}\rangle$, defined by the relations $\langle z^{+}|\varphi\rangle \equiv \overline{\langle\varphi|z^{+}\rangle}$ and $\langle\varphi|\widetilde{z}^{+}\rangle \equiv \overline{\langle\widetilde{z}^{+}|\varphi\rangle}$, satisfy

$$\langle z^{+}| = \langle\widetilde{\overline{z}}^{+}| \qquad |\widetilde{z}^{+}\rangle = |\overline{z}^{+}\rangle. \tag{26}$$

Since we assume that $R^{+}(z)$ has a simple pole at $z = z_0$ in the lower complex half plane, $R^{-}(z)$ has a simple pole at $z = \overline{z}_0$ in the upper complex half plane (see appendix B). Correspondingly, $|z^{+}\rangle$ has a pole at z_0 and $\langle\widetilde{z}^{+}|$ has a pole at $\overline{z}_0$.

In order to define the Gamov vectors we will make a contour deformation. Going back to equation (15), we can deform the integral path over $[0, +\infty)$ corresponding to the real variable ω' into the curve $C \cup \Gamma$ in the lower complex half plane (see figure 1). Taking into account the simple pole at z_0, the amplitude to find the state $\varphi(t)$ in the state ψ becomes

$$\begin{aligned} A(t) = \langle\psi|\exp(-\mathrm{i}Ht)|\varphi\rangle &= \oint_C \mathrm{d}z'\ \exp(-\mathrm{i}z't)\langle\psi|z'^{+}\rangle\langle\widetilde{z}'^{+}|\varphi\rangle \\ &\quad + \int_\Gamma \mathrm{d}z'\ \exp(-\mathrm{i}z't)\langle\psi|z'^{+}\rangle\langle\widetilde{z}'^{+}|\varphi\rangle \end{aligned} \tag{27}$$

and one obtains the equivalent well-defined equation

$$A(t) = \langle\psi|\exp(-\mathrm{i}Ht)|\varphi\rangle = \exp(-\mathrm{i}z_0t)\langle\psi|f_0\rangle\langle\widetilde{f}_0|\varphi\rangle + \int_\Gamma \mathrm{d}z'\ \exp(-\mathrm{i}z't)\langle\psi|f_{z'}\rangle\langle\widetilde{f}_{z'}|\varphi\rangle \tag{28}$$

where

$$\begin{aligned} &\langle\widetilde{f}_0|\varphi\rangle \equiv \mathrm{cont}_{\omega'\to z_0}\langle\omega'^{+}|\varphi\rangle \\ &\langle\psi|f_0\rangle \equiv (-2\pi\mathrm{i})\,\mathrm{cont}_{\omega'\to z_0}(\omega' - z_0)\langle\psi|\omega'^{+}\rangle \\ &\langle\widetilde{f}_{z'}|\varphi\rangle \equiv \mathrm{cont}_{\omega'\to z'}\langle\omega'^{+}|\varphi\rangle \\ &\langle\psi|f_{z'}\rangle \equiv \mathrm{cont}_{\omega'\to z'}\langle\psi|\omega'^{+}\rangle \qquad z' \in \Gamma. \end{aligned} \tag{29}$$

The complex conjugate amplitude is given by

$$\overline{A(t)} = \langle\varphi|\exp(\mathrm{i}Ht)|\psi\rangle = \exp(\mathrm{i}\overline{z}_0t)\langle\varphi|\widetilde{f}_0\rangle\langle f_0|\psi\rangle + \int_{\overline{\Gamma}} \mathrm{d}z\ \exp(+\mathrm{i}zt)\langle\varphi|\widetilde{f}_z\rangle\langle f_z|\psi\rangle \tag{30}$$

where

$$\begin{aligned} &\langle\varphi|\widetilde{f}_0\rangle \equiv \mathrm{cont}_{\omega\to\overline{z}_0}\langle\varphi|\omega^{+}\rangle \\ &\langle f_0|\psi\rangle \equiv (+2\pi\mathrm{i})\,\mathrm{cont}_{\omega\to\overline{z}_0}(\omega - \overline{z}_0)\langle\omega^{+}|\psi\rangle \\ &\langle\varphi|\widetilde{f}_z\rangle \equiv \mathrm{cont}_{\omega\to z}\langle\varphi|\omega^{+}\rangle \\ &\langle f_z|\psi\rangle \equiv \mathrm{cont}_{\omega\to z}\langle\omega^{+}|\psi\rangle \qquad z \in \overline{\Gamma}. \end{aligned} \tag{31}$$

From equations (29) and (31) it is easy to deduce the duality properties $\langle\widetilde{f}_0|\varphi\rangle = \overline{\langle\varphi|\widetilde{f}_0\rangle}$ and $\langle\psi|f_0\rangle = \overline{\langle f_0|\psi\rangle}$. Note that there is no duality relation between $|f_0\rangle$ and $\langle\widetilde{f}_0|$, nor between $|\widetilde{f}_0\rangle$ and $\langle f_0|$.

One can show (see appendix C) that the functionals defined in equations (29) and (31) are generalized eigenvectors of the Hamiltonian with complex eigenvalues, i.e.

$$\begin{array}{llllll} \langle\widetilde{f}_0|H = z_0\langle\widetilde{f}_0| & H|f_0\rangle = z_0|f_0\rangle & \langle\widetilde{f}_{z'}|H = z'\langle\widetilde{f}_{z'}| & H|f_{z'}\rangle = z'|f_{z'}\rangle & z' \in \Gamma. \\ H|\widetilde{f}_0\rangle = \overline{z}_0|\widetilde{f}_0\rangle & \langle f_0|H = \overline{z}_0\langle f_0| & H|\widetilde{f}_z\rangle = z|\widetilde{f}_z\rangle & \langle f_z|H = z\langle f_z| & z \in \overline{\Gamma}. \end{array} \tag{32}$$

Therefore the identity can be written as[12]

$$I = |f_0\rangle\langle\tilde{f}_0| + \int_\Gamma \mathrm{d}z\, |f_z\rangle\langle\tilde{f}_z|. \tag{33}$$

The generalized eigenvectors of H with the eigenvalues z_0 and $\overline{z}_0$, associated with the simple poles of the analytic extensions of the resolvent, are usually called 'Gamov vectors'.

It is important to note that while the amplitude $A(t)$ above is well defined in coordinate representation, the Gamov vectors diverge for growing values of the coordinates. For instance, in the case of a one-dimensional problem in $[0, +\infty)$ where the potential V has a compact support, one obtains[13]

$$\langle x|f_0\rangle \sim \exp(+\mathrm{i}\sqrt{z_0}x) \qquad \langle\widetilde{f}_0|x\rangle \sim \exp(+\mathrm{i}\sqrt{\overline{z}_0}x) \tag{34}$$

i.e. an oscillating function modulated by a growing exponential. Therefore, if one attempts to define the 'norm' of the functional $|f_0\rangle$ by $\langle f_0|f_0\rangle \equiv \int_0^\infty \mathrm{d}x\, \langle f_0|x\rangle\langle x|f_0\rangle$, the exponential growing integrand would give an infinite value. The energy $\langle f_0|H|f_0\rangle$ is also divergent and the internal product $\langle\widetilde{f}_0|f_0\rangle$ is not defined due to the oscillatory and diverging terms. These quantities are mathematically flawed since they are 'functionals of functionals'. In spite of this strong shortcoming they can be regularized and computed with the recipes quoted in the introduction. Since these methods suffer from the uncertainties that we have discussed above we will use a different approach. Expressions like $\langle\psi|f_0\rangle$ or $\langle\widetilde{f}_0|\varphi\rangle$ are generally well defined, at least for well-behaved 'test vectors' φ and ψ. For these test vectors, equation (28) gives a well-defined complex spectral decomposition of the transition amplitude $A(t)$. The survival amplitude can be obtained from equation (28) with $\varphi = \psi$. Moreover, if $|\mathrm{Im}\, z_0| \ll |\mathrm{Re}\, z_0|$, it can be proved that for intermediate values of time, the complex eigenvalue z_0 gives the main contribution to the survival probability of a pure state [25], i.e. $|\langle\varphi|\exp(-\mathrm{i}Ht)|\varphi\rangle|^2 \cong \exp(-\Gamma t)$, where $\Gamma \equiv 2\,|\mathrm{Im}\, z_0|$.

We have succeeded in defining Gamov vectors as functionals, but this is not enough to remove the problems associated with these vectors. Using equations (15), (28) and (30) we can write the probability $P(t)$ of finding the system in the pure state ψ at time t, if it was initially in the pure state φ

$$\begin{aligned} P(t) = \langle\varphi|\exp(+\mathrm{i}Ht)|\psi\rangle\langle\psi|\exp(-\mathrm{i}Ht)|\varphi\rangle \\ &= \exp[\mathrm{i}(\overline{z}_0 - z_0)t]\langle\varphi|\widetilde{f}_0\rangle\langle f_0|\psi\rangle\langle\psi|f_0\rangle\langle\widetilde{f}_0|\varphi\rangle \\ &\quad + \int_\Gamma \mathrm{d}z' \exp[\mathrm{i}(\overline{z}_0 - z')t]\langle\varphi|\widetilde{f}_0\rangle\langle f_0|\psi\rangle\langle\psi|f_{z'}\rangle\langle\widetilde{f}_{z'}|\varphi\rangle \\ &\quad + \int_{\overline{\Gamma}} \mathrm{d}z \exp[\mathrm{i}(z - z_0)t]\langle\varphi|\widetilde{f}_z\rangle\langle f_z|\psi\rangle\langle\psi|f_0\rangle\langle\widetilde{f}_0|\varphi\rangle \\ &\quad + \int_{\overline{\Gamma}} \mathrm{d}z \int_\Gamma \mathrm{d}z' \exp[\mathrm{i}(z - z')t]\langle\varphi|\widetilde{f}_z\rangle\langle f_z|\psi\rangle\langle\psi|f_{z'}\rangle\langle\widetilde{f}_{z'}|\varphi\rangle. \end{aligned} \tag{35}$$

The probability $P(t)$ is related to the projector $\Pi_\psi \doteq |\psi\rangle\langle\psi|$ by the expression

$$P(t) = \langle\varphi|\exp(+\mathrm{i}Ht)\Pi_\psi \exp(-\mathrm{i}Ht)|\varphi\rangle \tag{36}$$

[12] This is the partition of the identity within this formalism. It follows that $|f\rangle$ and $\langle\widetilde{f}|$ are the vectors of a 'biorthonormal' basis. From this structure it follows that the 'natural' norm is $\langle\widetilde{f}|f\rangle$, as in [9–11]. This solution deserves the criticism under equation (7).

[13] We consider the square roots for which $\mathrm{Im}(\sqrt{z_0}) < 0$ and $\mathrm{Im}(\sqrt{\overline{z}_0}) > 0$.

One is tempted to generalize to more general cases the expression given in equation (35), *which is valid to compute transition probabilities between normalized pure states*. For example, we may try to compute the probability of finding the particle at a distance greater than R at time t (this is equivalent to having detected the particle passing the point R before time t). To compute this probability using Gamov vectors, we may try to replace in equations (35) and (36) the projector $\Pi_\psi = |\psi\rangle\langle\psi|$ onto the pure state ψ, by the projector $\Pi_{[R,\infty)} \equiv \int_R^\infty \mathrm{d}x\, |x\rangle\langle x|$ onto a set of states localized at a distance greater than R, e.g. outside the potential barrier which produces the resonance (see also section 6). But then we find a new and unexpected problem: divergent terms appear. For instance $\langle f_0|\Pi_{[R,\infty)}|f_0\rangle = \int_R^\infty \mathrm{d}x \langle f_0|x\rangle\langle x|f_0\rangle = \infty$, due to the exponentially growing factor $\langle f_0|x\rangle\langle x|f_0\rangle \sim \exp(+\mathrm{i}[\sqrt{z_0} - \sqrt{\overline{z}_0}]x)$. The same kinds of troubles appear if one tries to compute the conserved total probability $1 = \langle\varphi_t|\varphi_t\rangle = \langle\varphi|\exp(+\mathrm{i}Ht)I\exp(-\mathrm{i}Ht)|\varphi\rangle$ by replacing the projector Π_ψ by $I = \int_0^\infty \mathrm{d}x\, |x\rangle\langle x|$ in equation (35).

We thus realize that the use of Gamov vectors to compute the time evolution of mean values for observables which are not simple projections onto a normalizable pure state, cannot be a straightforward generalization of the expression given in equation (35). This implies that if one wants to include resonances in the time evolution of observables a different approach is needed.

In the rest of this paper we will introduce a suitable formalism (cf [20, 21, 26]) to deal with general observables and to compute their time evolution using complex eigenvalues. We will also give a precise meaning to the 'energy' and the 'norm' of 'Gamov states' revealing their real nature.

3. Generalized states and observables

The expressions given in equations (9)–(11) for the operators I, H_0 and H, suggest that it is necessary to consider a general form for the self-adjoint operators representing observables of the system, namely

$$O = \int \mathrm{d}\omega\, O_\omega|\omega\rangle\langle\omega| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, O_{\omega\omega'}|\omega\rangle\langle\omega'| \tag{37}$$

where $O_\omega = \overline{O}_\omega$ and $O_{\omega\omega'} = \overline{O}_{\omega'\omega}$.

The first term in this equation can be written as $\int \mathrm{d}\omega \int \mathrm{d}\omega'\, O_\omega\delta(\omega-\omega')|\omega\rangle\langle\omega'|$. Since it contains a Dirac delta, we will call it the *singular term*. The second term has no singularity because $O_{\omega\omega'}$ is a regular function, and therefore we call this the *regular term*.

One cannot avoid the introduction of the singular term to get diagonal continuous matrices, e.g. to explain decoherence processes [22, 23]. For us terms like this will be essential in the formalism of sections 4 and 5. The most serious problem with the singular term is that often one faces situations where multiplications of Dirac deltas appear. It is to avoid such multiplications that the methods of [20, 21, 26] have been designed. However, in our formalism the Dirac deltas are avoided due to the introduction of the density operators. To see this let $|\psi_a\rangle$ be a pure state vector and p_a the probability of the quantum system to be in this pure state ($a = 1, 2, \ldots, \sum_a p_a = 1, \langle\psi_a|\psi_a\rangle = 1$). In this case, the state of the system can be represented by the density operator

$$\rho \equiv \sum_a p_a|\psi_a\rangle\langle\psi_a|. \tag{38}$$

The mean value of an observable represented by an operator O of the form given in equation (37) is

$$\langle O\rangle_\rho = \mathrm{Tr}(\rho O) = \int \mathrm{d}\omega \left[\sum_a p_a \langle\omega|\psi_a\rangle\langle\psi_a|\omega\rangle\right] O_\omega$$
$$+ \int \mathrm{d}\omega \int \mathrm{d}\omega' \left[\sum_a p_a \langle\omega'|\psi_a\rangle\langle\psi_a|\omega\rangle\right] O_{\omega\omega'}. \tag{39}$$

Defining

$$\rho_\omega \equiv \sum_a p_a \langle\omega|\psi_a\rangle\langle\psi_a|\omega\rangle \qquad \rho_{\omega\omega'} \equiv \sum_a p_a \langle\omega'|\psi_a\rangle\langle\psi_a|\omega\rangle \tag{40}$$

the mean value of the operator O in the state ρ can be written in the compact form

$$\langle O\rangle_\rho = \int \mathrm{d}\omega\, \rho_\omega O_\omega + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \rho_{\omega\omega'} O_{\omega\omega'}. \tag{41}$$

But from a more general point of view, ρ_ω and $\rho_{\omega\omega'}$ can be considered as the 'components' of a *linear functional* $(\rho|$, acting on the observable $|O)$ which is defined by its own 'components' O_ω and $O_{\omega\omega'}$. As we will see, in this way we can define 'generalized states' that contain the density operator of equation (38) as a particular case. The action of the state functional on the observable provides the mean value $\langle O\rangle_\rho = (\rho|O)$. In this approach, it is convenient to define the 'generalized observables'

$$|\omega) \equiv |\omega\rangle\langle\omega| \qquad |\omega\omega') \equiv |\omega\rangle\langle\omega'| \tag{42}$$

in such a way that the observable O can be written as

$$|O) \equiv O = \int \mathrm{d}\omega\, O_\omega|\omega) + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, O_{\omega\omega'}|\omega\omega') \tag{43}$$

and therefore $(|\omega), |\omega,\omega'))$ is the basis of the space of the observables. It is also useful to define the generalized states $(\tilde{\omega}|$ and $(\widetilde{\omega\omega'}|$ satisfying the relations

$$(\tilde{\omega}|O) \equiv O_\omega \qquad (\widetilde{\omega\omega'}|O) \equiv O_{\omega\omega'}. \tag{44}$$

It should be emphasized that according to these definitions, $(\tilde{\omega}| \neq (|\omega\rangle\langle\omega|)^\dagger$ and $(\widetilde{\omega\omega'}| \neq (|\omega\rangle\langle\omega'|)^\dagger$, in contrast to the case of discrete spectra. Using the generalized states defined in equations (44), the state functional reads

$$(\rho| = \int \mathrm{d}\omega\, \rho_\omega(\tilde{\omega}| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \rho_{\omega\omega'}(\widetilde{\omega\omega'}|. \tag{45}$$

The generalized states $(\tilde{\omega}|$ and $(\widetilde{\omega\omega'}|$ form a basis for the dual of the observable space, namely the state space.

The generalized states $(\tilde{\omega}|, (\widetilde{\omega\omega'}|$ and observables $|\omega), |\omega\omega')$ form a complete biorthonormal system to describe observables and states of the form given in equations (43) and (45). It is straightforward to verify the orthogonality and completeness conditions

$$(\tilde{\omega}|\omega') = \delta(\omega-\omega') \qquad (\widetilde{\omega\omega'}|\varepsilon\varepsilon') = \delta(\omega-\varepsilon)\delta(\omega'-\varepsilon') \qquad (\tilde{\omega}|\varepsilon\varepsilon') = (\widetilde{\omega\omega'}|\varepsilon) = 0 \tag{46}$$

$$(\rho|\mathbb{I} = (\rho|, \quad \mathbb{I}|O) = |O) \tag{47}$$

where $\mathbb{I}$ is the *identity superoperator* given by

$$\mathbb{I} \equiv \int \mathrm{d}\omega\, |\omega)(\tilde{\omega}| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, |\omega\omega')(\widetilde{\omega\omega'}| \tag{48}$$

which of course differs from the *identity operator* $I = \int \mathrm{d}\omega\, |\omega\rangle\langle\omega|$.

Up to this point, we have only provided an alternative mathematical framework for the description of states that can be also described in terms of the well-known density operator given in equation (38). However, we will show in what follows that the spectral decomposition of the time evolution of a quantum system with continuous spectrum includes *generalized states* which are functionals like those of equation (45), and which cannot be described by the usual density operators. The well-known conditions $\mathrm{Tr}\,\rho = 1$ and $\rho^\dagger = \rho$ for the density operator, must be replaced in this formalism by the conditions of *total probability* and *reality* on the state functionals (see [20] for details)

$$(\rho|I) = 1 \qquad (\rho|O^\dagger) = \overline{(\rho|O)}. \tag{49}$$

From these two conditions one learns that the components of the states should satisfy $\int \mathrm{d}\omega\, \rho_\omega = 1$, $\rho_\omega = \overline{\rho_\omega}$ and $\rho_{\omega'\omega} = \overline{\rho_{\omega\omega'}}$. The *positivity* condition remains the usual one, i.e. $\rho_\omega = \overline{\rho_\omega} \geqslant 0$.

The time evolution of the state functionals is obtained from the equation

$$(\rho_t|O) \equiv (\rho_0|O_t) = (\rho_0|\exp(+\mathrm{i}Ht)O\exp(-\mathrm{i}Ht)) \tag{50}$$

relating Schrödinger and Heisenberg representations.

4. Generalized real spectral decomposition of the time evolution

If we compute the matrix elements of an operator O of the form given in equation (37) in the H_0 representation, we obtain $\langle\omega|O|\omega'\rangle = \delta(\omega-\omega')O_\omega + O_{\omega\omega'}$ (where the singular and the regular terms naturally appear). If one uses the H representation, the matrix element $\langle\omega^+|O|\omega'^+\rangle$ also include a singular term $\delta(\omega-\omega')O_\omega$, and therefore there is a term of the form $\int \mathrm{d}\omega\, O_\omega|\omega^+\rangle\langle\omega^+|$ in the operator O. This term is time independent in the Heisenberg representation.

Since Gamov vectors are exponentially decaying states they cannot be contained in the time-independent term. We therefore separate the time independent from the time-dependent part of the observable, hoping to find the Gamov vectors in the time-dependent part. To do this we define the invariant and the non-invariant or 'fluctuating' parts of O as follows

$$O_{\mathrm{inv}} \doteq \int \mathrm{d}\omega\, O_\omega|\omega^+\rangle\langle\omega^+| \qquad O_{\mathrm{fluc}} \doteq O - O_{\mathrm{inv}}. \tag{51}$$

The matrix elements of O_{fluc} are

$$\begin{aligned}\langle\omega^+|O_{\mathrm{fluc}}|\omega'^+\rangle = &\int \mathrm{d}\varepsilon\,[\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\varepsilon-\omega')](\tilde{\varepsilon}|O)\\ &+ \int \mathrm{d}\varepsilon \int \mathrm{d}\varepsilon'\,\langle\omega^+|\varepsilon\rangle\langle\varepsilon'|\omega'^+\rangle(\widetilde{\varepsilon\varepsilon'}|O).\end{aligned} \tag{52}$$

The time-dependent contribution to the mean value is given by the fluctuating part since it is

$$\begin{aligned}\langle O\rangle_t &= (\rho_0|\,\mathrm{e}^{+\mathrm{i}Ht}O\,\mathrm{e}^{-\mathrm{i}Ht})\\ &= (\rho_0|O_{\mathrm{inv}}) + \left(\rho_0|\,\mathrm{e}^{+\mathrm{i}Ht}O_{\mathrm{fluc}}\,\mathrm{e}^{-\mathrm{i}Ht}\right)\\ &= \int \mathrm{d}\omega\,(\rho_0\|\omega^+\rangle\langle\omega^+|)O_\omega + \int \mathrm{d}\omega \int \mathrm{d}\omega'\,\mathrm{e}^{\mathrm{i}(\omega-\omega')t}(\rho_0\|\omega^+\rangle\langle\omega'^+|)\langle\omega^+|O_{\mathrm{fluc}}|\omega'^+\rangle.\end{aligned} \tag{53}$$

Defining the following *generalized* states and observables

$$
\begin{aligned}
&|\Phi_\omega) \equiv |\omega^+\rangle\langle\omega^+| \\
&(\widetilde{\Phi}_\omega| \equiv (\tilde{\omega}| \\
&|\Phi_{\omega\omega'}) \equiv |\omega^+\rangle\langle\omega'^+| \\
&(\widetilde{\Phi}_{\omega\omega'}| \equiv \int \mathrm{d}\varepsilon\, [\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\varepsilon-\omega')](\tilde{\varepsilon}| + \int \mathrm{d}\varepsilon \int \mathrm{d}\varepsilon'\, \langle\omega^+|\varepsilon\rangle\langle\varepsilon'|\omega'^+\rangle(\widetilde{\varepsilon\varepsilon'}|
\end{aligned}
\tag{54}
$$

and using equations (51)–(54), one gets the compact form

$$
\langle O\rangle_t = (\rho_t|O) = \int \mathrm{d}\omega\, (\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|O) + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \mathrm{e}^{\mathrm{i}(\omega-\omega')t}(\rho_0|\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}|O). \tag{55}
$$

From this last expression we can obtain the time dependence of the state functional (in the Schrödinger representation):

$$
(\rho_t| = \int \mathrm{d}\omega\, (\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \mathrm{e}^{\mathrm{i}(\omega-\omega')t}(\rho_0|\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}|. \tag{56}
$$

In the next section we will see that the Gamov states are contained in the analytic continuation of the last term.

The generalized states and observables defined in equations (54) have interesting properties:

(i) *They form a complete biorthogonal system for observables and states*: this is rather straightforward, since

$$
\begin{aligned}
&(\widetilde{\Phi}_\omega|\Phi_{\omega'}) = \delta(\omega-\omega') \qquad (\widetilde{\Phi}_{\omega\omega'}|\Phi_{\varepsilon\varepsilon'}) = \delta(\omega-\varepsilon)\delta(\omega'-\varepsilon') \\
&\qquad (\widetilde{\Phi}_\omega|\Phi_{\varepsilon\varepsilon'}) = (\widetilde{\Phi}_{\varepsilon\varepsilon'}|\Phi_{\omega'}) = 0.
\end{aligned}
\tag{57}
$$

The identity superoperator $\mathbb{I}$, already defined in equation (48), can be written in the form

$$
\mathbb{I} = \int \mathrm{d}\omega\, |\Phi_\omega)(\widetilde{\Phi}_\omega| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, |\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}|. \tag{58}
$$

(ii) *They provide the spectral decomposition of the time evolution generator*: in the Heisenberg representation the time evolution of an observable O of the form given in equation (37) is given by $O_t = \exp(+\mathrm{i}Ht)O\exp(-\mathrm{i}Ht) = \exp(+\mathrm{i}\mathbb{L}t)O$, where $\mathbb{L}$ is the Liouville–von Neumann superoperator, defined by $\mathbb{L}O \equiv HO - OH$. It is

$$
\mathbb{L} = \int \mathrm{d}\omega \int \mathrm{d}\omega'\, (\omega-\omega')|\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}|. \tag{59}
$$

Therefore $|\Phi_\omega)$ $((\widetilde{\Phi}_\omega|)$ is a right (left) eigenvector of $\mathbb{L}$ with zero eigenvalue, and $|\Phi_{\omega\omega'})$ $((\widetilde{\Phi}_{\omega\omega'}|)$ is a right (left) eigenvector of $\mathbb{L}$ with eigenvalue $(\omega-\omega')$. Gamov states will also be eigenvectors of $\mathbb{L}$ but with complex eigenvalues.

(iii) *The generalized states* $(\widetilde{\Phi}_\omega|$ *and* $\widetilde{\Phi}_{\omega\omega'}|$ *have well-defined physical properties*: any state functional can be written as the linear combination

$$
(\rho| = (\rho|\mathbb{I} = \int \mathrm{d}\omega\, (\rho|\Phi_\omega)(\widetilde{\Phi}_\omega| + \int \mathrm{d}\omega \int \mathrm{d}\omega'\, (\rho|\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}| \tag{60}
$$

and therefore $(\widetilde{\Phi}_\omega|$ and $(\widetilde{\Phi}_{\omega\omega'}|$ can be considered as a basis of generalized states.

The generalized state $(\widetilde{\Phi}_\omega|$ satisfies

$$
\begin{aligned}
(\widetilde{\Phi}_\omega|I) &= (\widetilde{\Phi}_\omega| \int \mathrm{d}\omega'\, |\omega'\rangle\langle\omega'|) = (\widetilde{\Phi}_\omega| \int \mathrm{d}\omega'\, |\omega') \\
&= \int \mathrm{d}\omega'\, (\omega|\omega') = \int \mathrm{d}\omega' \delta(\omega-\omega') = 1
\end{aligned}
\tag{61}
$$

$$(\widetilde{\Phi}_\omega|H) = (\widetilde{\Phi}_\omega|\left[\int \mathrm{d}\omega'\,\omega'|\omega') + \int \mathrm{d}\omega' \int \mathrm{d}\omega''\,V_{\omega'\omega''}|\omega'\omega'')\right]$$
$$= \int \mathrm{d}\omega'\,\omega'(\omega|\omega') + \int \mathrm{d}\omega' \int \mathrm{d}\omega''\,V_{\omega'\omega''}(\omega|\omega'\omega'')$$
$$= \int \mathrm{d}\omega'\,\omega'\delta(\omega-\omega') = \omega. \tag{62}$$

Therefore $(\widetilde{\Phi}_\omega|$ verifies the total probability condition $(\widetilde{\Phi}_\omega|I) = 1$ (the generalization of the condition $\mathrm{Tr}\rho = 1$ for the usual density operators). The mean value of the energy is $\langle H\rangle = (\widetilde{\Phi}_\omega|H) = \omega$. Moreover, one can show that $\langle H^n\rangle = \left(\widetilde{\Phi}_\omega|H^n\right) = \omega^n$ $(n = 1, 2, \ldots)$, which implies $\langle (H - \langle H\rangle)^n\rangle = 0$. In summary, the mean value ω of the energy has no dispersion, and one can say that the state $(\widetilde{\Phi}_\omega|$ *has* energy ω. It is clear from the definition that this is a generalized state which cannot be represented, neither by a normalized wavefunction nor by a density operator.

The generalized state $(\widetilde{\Phi}_{\omega\omega'}|$ satisfies

$$(\widetilde{\Phi}_{\omega\omega'}|I) = (\widetilde{\Phi}_{\omega\omega'}|\int \mathrm{d}\varepsilon'|\varepsilon')$$
$$= \int \mathrm{d}\varepsilon\,[\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\varepsilon-\omega')](\varepsilon|\int \mathrm{d}\varepsilon'\,|\varepsilon')$$
$$= \int \mathrm{d}\varepsilon\,[\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\varepsilon-\omega')]$$
$$= \int \mathrm{d}\varepsilon\,\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \int \mathrm{d}\varepsilon\,\delta(\omega-\varepsilon)\delta(\varepsilon-\omega')$$
$$= \delta(\omega-\omega') - \delta(\omega-\omega') = 0. \tag{63}$$

$$(\widetilde{\Phi}_{\omega\omega'}|H) = \int \mathrm{d}\varepsilon\,[\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\varepsilon-\omega')]\varepsilon$$
$$+ \int \mathrm{d}\varepsilon \int \mathrm{d}\varepsilon'\,\langle\omega^+|\varepsilon\rangle\langle\varepsilon'|\omega'^+\rangle V_{\varepsilon\varepsilon'}$$
$$= \langle\omega^+|\left[\int \mathrm{d}\varepsilon\,\varepsilon|\varepsilon\rangle\langle\varepsilon| + \int \mathrm{d}\varepsilon \int \mathrm{d}\varepsilon'\,V_{\varepsilon\varepsilon'}|\varepsilon\rangle\langle\varepsilon'|\right]|\omega'^+\rangle - \delta(\omega-\omega')\omega$$
$$= \langle\omega^+|H|\omega'^+\rangle - \delta(\omega-\omega')\omega = 0. \tag{64}$$

The generalized states $(\widetilde{\Phi}_{\omega\omega'}|$ of the real spectral decomposition satisfy $(\widetilde{\Phi}_{\omega\omega'}|I) = 0$ and $(\widetilde{\Phi}_{\omega\omega'}|H) = 0$, i.e. they have zero values of the energy and the generalized trace. However, the $(\widetilde{\Phi}_{\omega\omega'}|$ are very important, because they provide the time-dependent part of any physical state functional (see equations (55) and (56)). Gamov vectors will inherit these rigorous mathematical properties as we will see in the next section. This is the clue that will yield the real nature of Gamov states. The energy and the non-zero part of the generalized trace is carried out by the time-independent components $(\widetilde{\Phi}_\omega|$, satisfying $(\widetilde{\Phi}_\omega|H) = \omega$ and $(\widetilde{\Phi}_\omega|I) = 1$. Therefore both $(\widetilde{\Phi}_\omega|$ and $(\widetilde{\Phi}_{\omega\omega'}|$ components are physically relevant parts of a state functional.

(iv) *At very long times, i.e. asymptotically in time, they provide a suitable representation*: if $(\rho_0|\Phi_{\omega\omega'})$ and $(\widetilde{\Phi}_{\omega\omega'}|O)$ are regular functions of the variables ω and ω', the second factor of the expression given in equation (55) for the time-dependent mean value $\langle O\rangle_t$ of the observable tends to vanish for very long times, due to the rapidly oscillating factor $\mathrm{e}^{\mathrm{i}(\omega-\omega')t}$

inside the double integral. Therefore, we obtain $\lim_{t\to\infty}(\rho_t|O) = \int \mathrm{d}\omega\,(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|O)$, or (in the weak sense)

$$(\rho_\infty| \equiv W \lim_{t\to\infty}(\rho_t| = \int \mathrm{d}\omega\,(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|. \tag{65}$$

Therefore the components $(\widetilde{\Phi}_{\omega\omega'}|$ of the state are eliminated during the time evolution, and the properties $(\widetilde{\Phi}_{\omega\omega'}|H) = 0$ and $(\widetilde{\Phi}_{\omega\omega'}|I) = 0$, discussed above, are now found to be *essential* for energy and probability conservation, i.e.

$$\langle H\rangle = (\rho_0|H) = (\rho_t|H) = (\rho_\infty|H) \qquad \langle I\rangle = (\rho_0|I) = (\rho_t|I) = (\rho_\infty|I) = 1. \tag{66}$$

5. Generalized complex spectral decomposition of the time evolution of observables and Gamov states

We obtained in equation (55) the spectral decomposition of the mean value of an observable, i.e.

$$\langle O\rangle_t = (\rho_t|O) = \int_0^\infty \mathrm{d}\omega\,(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|O) + \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega'\,\mathrm{e}^{\mathrm{i}(\omega-\omega')t}(\rho_0|\Phi_{\omega\omega'})(\widetilde{\Phi}_{\omega\omega'}|O). \tag{67}$$

We wish to deform the integral over $[0, +\infty)$ for the variable ω (ω') into a curve in the upper (lower) complex half plane. Therefore we need the following analytic extensions

$$(\rho_0|\Phi_{zz'}) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z'}(\rho_0|\Phi_{\omega\omega'}) \tag{68}$$

$$(\widetilde{\Phi}_{zz'}|O) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z'}(\widetilde{\Phi}_{\omega\omega'}|O). \tag{69}$$

Equations (25) and (54) and the assumption of a simple pole for the analytic extension of the resolvent to the lower complex half plane, show that if $z \in \mathbb{C}^+$ and $z' \in \mathbb{C}^-$, $(\rho_0|\Phi_{zz'})$ is analytic and $(\widetilde{\Phi}_{zz'}|O)$ has simple poles for $z = \overline{z}_0$ and $z' = z_0$. It is therefore possible to deform the integrals over the real variables ω and ω' to the curves indicated in figures 2 and 1, respectively. The following expression is obtained for the time evolution of the mean value

$$\begin{aligned}(\rho_t|O) = \int_0^\infty &\mathrm{d}\omega\,(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|O) + \mathrm{e}^{\mathrm{i}(\overline{z}_0-z_0)t}(\rho_0|\Phi_{00})(\widetilde{\Phi}_{00}|O)\\ &+ \int_\Gamma \mathrm{d}z'\,\mathrm{e}^{\mathrm{i}(\overline{z}_0-z')t}(\rho_0|\Phi_{0z'})(\widetilde{\Phi}_{0z'}|O) + \int_{\overline{\Gamma}} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}(z-z_0)t}(\rho_0|\Phi_{z0})(\widetilde{\Phi}_{z0}|O)\\ &+ \int_{\overline{\Gamma}} \mathrm{d}z \int_\Gamma \mathrm{d}z'\,\mathrm{e}^{\mathrm{i}(z-z')t}(\rho_0|\Phi_{zz'})(\widetilde{\Phi}_{zz'}|O)\end{aligned} \tag{70}$$

where equation (67) was used and the following functionals were introduced:

$$\begin{aligned}
&(\rho_0|\Phi_{00}) \equiv \mathrm{cont}_{\omega\to\overline{z}_0}\,\mathrm{cont}_{\omega'\to z_0}(\rho_0|\Phi_{\omega\omega'}) = (\rho_0\|\widetilde{f}_0\rangle\langle\widetilde{f}_0|)\\
&(\widetilde{\Phi}_{00}|O) \equiv \mathrm{cont}_{\omega\to\overline{z}_0}\,\mathrm{cont}_{\omega'\to z_0}4\pi^2(\omega-\overline{z}_0)(\omega'-z_0)(\widetilde{\Phi}_{\omega\omega'}|O) = \langle f_0|(O-O_{\mathrm{inv}})|f_0\rangle\\
&(\rho_0|\Phi_{0z'}) \equiv \mathrm{cont}_{\omega\to\overline{z}_0}\,\mathrm{cont}_{\omega'\to z'}(\rho_0|\Phi_{\omega\omega'}) = (\rho_0\|\widetilde{f}_0\rangle\langle\widetilde{f}_{z'}|)\\
&(\widetilde{\Phi}_{0z'}|O) \equiv \mathrm{cont}_{\omega\to\overline{z}_0}\,\mathrm{cont}_{\omega'\to z'}(2\pi\,\mathrm{i})(\omega-\overline{z}_0)(\widetilde{\Phi}_{\omega\omega'}|O) = \langle f_0|(O-O_{\mathrm{inv}})|f_{z'}\rangle\\
&(\rho_0|\Phi_{z0}) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z_0}(\rho_0|\Phi_{\omega\omega'}) = (\rho_0\|\widetilde{f}_z\rangle\langle\widetilde{f}_0|)\\
&(\widetilde{\Phi}_{z0}|O) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z_0}(-2\pi\,\mathrm{i})(\omega'-z_0)(\widetilde{\Phi}_{\omega\omega'}|O) = \langle f_z|(O-O_{\mathrm{inv}})|f_0\rangle\\
&(\rho_0|\Phi_{zz'}) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z'}(\rho_0|\Phi_{\omega\omega'}) = (\rho_0\|\widetilde{f}_z\rangle\langle\widetilde{f}_{z'}|)\\
&(\widetilde{\Phi}_{zz'}|O) \equiv \mathrm{cont}_{\omega\to z}\,\mathrm{cont}_{\omega'\to z'}(\widetilde{\Phi}_{\omega\omega'}|O) = \langle f_z|(O-O_{\mathrm{inv}})|f_{z'}\rangle.
\end{aligned} \tag{71}$$

It is important to note that in the definitions of these functionals the analytic continuations should be understood in the weak sense, i.e. they must be performed *after* the application of the functionals depending on the real parameters ω and ω' to suitable test functions. This is clear from the fact that the new spectral decomposition given in equation (70) was obtained using the Cauchy theorem in equation (67).

From equation (70) we obtain the time dependence of a state functional through the complex spectral decomposition

$$(\rho_t| = \int_0^\infty \mathrm{d}\omega\,(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega| + \mathrm{e}^{\mathrm{i}(\overline{z}_0 - z_0)t}(\rho_0|\Phi_{00})(\widetilde{\Phi}_{00}| + \int_{\overline{\Gamma}} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}(z-z_0)t}(\rho_0|\Phi_{z0})(\widetilde{\Phi}_{z0}| + \int_{\overline{\Gamma}} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}(z-z_0)t}(\rho_0|\Phi_{z0})(\widetilde{\Phi}_{z0}| + \int_{\overline{\Gamma}} \mathrm{d}z \int_{\Gamma} \mathrm{d}z'\,\mathrm{e}^{\mathrm{i}(z-z')t}(\rho_0|\Phi_{zz'})(\widetilde{\Phi}_{zz'}|. \qquad (72)$$

Equations (70) and (72) provide an alternative spectral decomposition to that given by equation (67), where the resonances at z_0 and $\overline{z}_0$ explicitly appear. Since $\overline{z}_0 - z_0 = -2\mathrm{i}\,\mathrm{Im}\,z_0$, and by definition $\mathrm{Im}\,z_0 < 0$, $(\widetilde{\Phi}_{00}|$ is an exponentially decaying mode and therefore a *generalized Gamov state*. This decomposition has the same properties as that in the previous section, namely[14]

(i) *They form a basis for observables and states*: the identity superoperator $\mathbb{I}$ can be written in the form

$$\mathbb{I} = \int \mathrm{d}\omega\,|\Phi_\omega)(\widetilde{\Phi}_\omega| + |\Phi_{00})(\widetilde{\Phi}_{00}| + \int_\Gamma \mathrm{d}z'\,|\Phi_{0z'})(\widetilde{\Phi}_{0z'}| + \int_{\overline{\Gamma}} \mathrm{d}z\,|\Phi_{z0})(\widetilde{\Phi}_{z0}| + \int_\Gamma \mathrm{d}z \int_{\Gamma'} \mathrm{d}z'\,|\Phi_{zz'})(\widetilde{\Phi}_{zz'}|. \qquad (74)$$

(ii) *They provide the spectral decomposition of the time evolution generator*:

$$\mathbb{L} = (\overline{z}_0 - z_0)|\Phi_{00})(\widetilde{\Phi}_{00}| + (\overline{z}_0 - z') \int_\Gamma \mathrm{d}z'\,|\Phi_{0z'})(\widetilde{\Phi}_{0z'}| + \int_{\overline{\Gamma}} \mathrm{d}z\,(z - z_0)|\Phi_{z0})(\widetilde{\Phi}_{z0}| + \int_{\overline{\Gamma}} \mathrm{d}z \int_\Gamma \mathrm{d}z'\,(z - z')|\Phi_{zz'})(\widetilde{\Phi}_{zz'}|. \qquad (75)$$

Therefore $|\Phi_{00})$ $((\widetilde{\Phi}_{00}|)$ is a right (left) eigenvector of $\mathbb{L}$ with eigenvalue $\overline{z}_0 - z_0 = -2\mathrm{i}\,\mathrm{Im}\,z_0$, $|\Phi_{0z'})((\widetilde{\Phi}_{0z'}|)$ is a right (left) eigenvector of $\mathbb{L}$ with eigenvalue $(\overline{z}_0 - z')$, $|\Phi_{z0})$ $((\widetilde{\Phi}_{z0}|)$ is a right (left) eigenvector of $\mathbb{L}$ with eigenvalue $(z - z_0)$, and $|\Phi_{zz'})$ $((\widetilde{\Phi}_{zz'}|)$ is a right (left) eigenvector of $\mathbb{L}$ with eigenvalue $(z - z')$. Gamov state $(\widetilde{\Phi}_{00}|$ will give the exponentially decaying term of the evolution.

(iii) *The generalized states have well-defined physical properties*: we have proved in the previous section that $(\widetilde{\Phi}_{\omega\omega'}|H) = (\widetilde{\Phi}_{\omega\omega'}|I) = 0$. This property is also verified by the new generalized states $(\widetilde{\Phi}_{00}|$, $(\widetilde{\Phi}_{0z'}|$, $(\widetilde{\Phi}_{z0}|$ and $(\widetilde{\Phi}_{zz'}|$, as they are obtained by analytic extensions of the functional $(\widetilde{\Phi}_{\omega\omega'}|$ (the analytic extension of zero is zero!)[15]. In spite of the fact that these functionals have zero value for the energy and the generalized trace, they are physically relevant because they expand the time-dependent part of any physical state in the complex spectral decomposition (see equations (70) and (72))

(iv) *They provide a suitable representation to describe the asymptotic (in time) behaviour of a state*: as in the previous section, only the first term of equation (72) remains when $t \to \infty$, and we also obtain $W \lim_{t\to\infty}(\rho_t| = \int \mathrm{d}\omega(\rho_0|\Phi_\omega)(\widetilde{\Phi}_\omega|$.

[14] It can be proved that

$$(\widetilde{\Phi}_\omega|\Phi_{\omega'}) = \delta(\omega - \omega') \qquad (\widetilde{\Phi}_{00}|\Phi_{00}) = 1 \qquad (\widetilde{\Phi}_\omega|\Phi_{00}) = (\widetilde{\Phi}_{00}|\Phi_{\omega'}) = 0, \text{etc.} \qquad (73)$$

In particular $(\widetilde{\Phi_{00}}|\Phi_{00}) = 1$ implies $\langle \tilde{f}_0|f_0\rangle = 1$, which are the usual results quoted in the literature. But we prefer not to use these and similar equations because they are not rigorous and are indeed unnecessary.

[15] The rigorous property $(\widetilde{\Phi}_{00}|I) = 0$ substitutes in our formalism for the dubious one $\langle f_0|f_0\rangle = 0$.

6. Application: one-dimensional potential barrier

In this section we will apply the formalism presented above to a realistic case. We will first derive general expressions for the probability of a particle to penetrate a barrier. We thus consider an initially pure state $|\varphi\rangle$ localized inside the potential barrier. The probability to find the particle at a distance larger than a given value R at time t is given by $(\rho_t|\Pi_{[R,\infty)})$, where

$$\Pi_{[R,\infty)} = \int_R^\infty \mathrm{d}x\, |x\rangle\langle x| = \int_0^\infty \mathrm{d}x\, |x\rangle\langle x| - \int_0^R \mathrm{d}x\, |x\rangle\langle x|. \tag{76}$$

In this expression, the vector $|x\rangle$ represents an eigenvector of the coordinate operator with eigenvalue x. Comparing with the decomposition $O = O_{\text{inv}} + O_{\text{fluc}}$ given in equation (51) we obtain

$$\Pi^{\text{inv}}_{[R,\infty)} = I = \int \mathrm{d}\omega\, |\omega^+\rangle\langle\omega^+| = \int \mathrm{d}\omega\, |\omega\rangle\langle\omega| = \int \mathrm{d}x\, |x\rangle\langle x|$$
$$\Pi^{\text{fluc}}_{[R,\infty)} = -\int_0^R \mathrm{d}x\, |x\rangle\langle x| \tag{77}$$

and therefore

$$(\widetilde{\Phi}_\omega|\Pi_{[R,\infty)}) = (\omega|\Pi_{[R,\infty)}) = 1$$
$$(\widetilde{\Phi}_{\omega\omega'}|\Pi_{[R,\infty)}) = \langle\omega^+|\Pi^{\text{fluc}}_{[R,\infty)}|\omega'^+\rangle = -\int_0^R \mathrm{d}x\, \langle\omega^+|x\rangle\langle x|\omega'^+\rangle. \tag{78}$$

Initially the system is assumed to be in a pure state $\rho_0 = |\varphi\rangle\langle\varphi|$, and therefore $(\rho_0|O) = \langle\varphi|O|\varphi\rangle$. One therefore has

$$\begin{aligned}(\rho_0|\Phi_\omega) &= (\rho_0\||\omega^+\rangle\langle\omega^+|) = \langle\varphi|\omega^+\rangle\langle\omega^+|\varphi\rangle\\ (\rho_0|\Phi_{\omega\omega'}) &= (\rho_0\||\omega^+\rangle\langle\omega'^+|) = \langle\varphi|\omega^+\rangle\langle\omega'^+|\varphi\rangle.\end{aligned} \tag{79}$$

Replacing equations (78) and (79) in the expression given in equation (55) for the time evolution of the mean value, we obtain

$$\begin{aligned}(\rho(t)|\Pi_{[R,\infty)}) &= \langle\Pi_{[R,\infty)}\rangle_t\\ &= 1 - \int_0^R \mathrm{d}x \int \mathrm{d}\omega \int \mathrm{d}\omega'\, \exp[\mathrm{i}(\omega-\omega')t]\langle\varphi|\omega^+\rangle\langle\omega^+|x\rangle\langle x|\omega'^+\rangle\langle\omega'^+|\varphi\rangle.\end{aligned} \tag{80}$$

If for $t = 0$ the particle is located at $x < R$ this last expression satisfies

$$\langle\Pi_{[R,\infty)}\rangle_{t=0} = 0 \qquad \lim_{t\to\infty}\langle\Pi_{[R,\infty)}\rangle_t = 1 \tag{81}$$

i.e. the probability of finding the particle at $x > R$ and $t = 0$ is zero and it will be 1 asymptotically, i.e. for large values of t.

Since the transition of the probability $\langle\Pi_{[R,\infty)}\rangle_t$ from zero to 1 may be dominated by an exponential behaviour, it may be convenient to use the complex spectral decomposition given in equation (70)

$$\begin{aligned}\langle\Pi_{[R,\infty)}\rangle_t = 1 &- \mathrm{e}^{\mathrm{i}(\overline{z}_0-z_0)t}\langle\varphi|\widetilde{f}_0\rangle\langle\widetilde{f}_0|\varphi\rangle \int_0^R \mathrm{d}x\, \langle f_0|x\rangle\langle x|f_0\rangle\\ &- \int_\Gamma \mathrm{d}z'\, \mathrm{e}^{\mathrm{i}(\overline{z}_0-z')t}\langle\varphi|\widetilde{f}_0\rangle\langle\widetilde{f}_{z'}|\varphi\rangle \int_0^R \mathrm{d}x\, \langle f_0|x\rangle\langle x|f_{z'}\rangle\\ &- \int_{\overline{\Gamma}} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}(z-z_0)t}\langle\varphi|\widetilde{f}_z\rangle\langle\widetilde{f}_0|\varphi\rangle \int_0^R \mathrm{d}x\, \langle f_z|x\rangle\langle x|f_0\rangle\\ &- \int_{\overline{\Gamma}} \mathrm{d}z \int_\Gamma \mathrm{d}z'\, \mathrm{e}^{\mathrm{i}(z-z')t}\langle\varphi|\widetilde{f}_z\rangle\langle\widetilde{f}_{z'}|\varphi\rangle \int_0^R \mathrm{d}x\, \langle f_z|x\rangle\langle x|f_{z'}\rangle.\end{aligned} \tag{82}$$

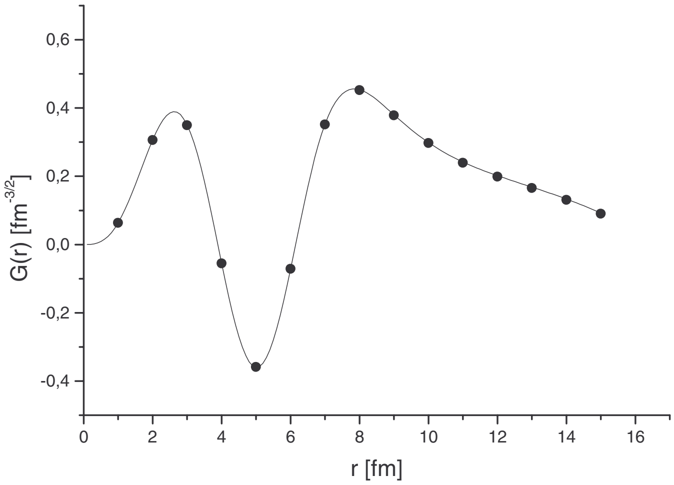

Figure 3. Real part of the Gamov function $G(r)$ versus the radius corresponding to the potential discussed in the text. The dots are the values calculated within our formalism.

This is a well-defined expression, free of any divergent term and fully within the framework of quantum mechanics. Yet, it contains all the advantages of treatments where the Gamov vectors are included [13, 14, 17].

It is interesting to see whether the diverging Gamov function, normalized according to the prescriptions that have been proposed so far (see, e.g., [6, 9, 10]), coincides with the one evaluated according to our formalism, i.e. equation (29). To do this we choose a realistic case, namely a $^{208}\mathrm{Pb}(2\mathrm{d}_{5/2})$ proton state in a Woods–Saxon potential, including spin–orbit interaction, with parameters as in [17], except the depth of the potential which we choose to be $V_0 = 60$ MeV. To calculate the corresponding Gamov state we used the computer code of [27]. We obtained for the energy of this state the value $z_0 = (9.940 - \mathrm{i}\,0.150)$ MeV, i.e. a width of $\Gamma = 300\,\mathrm{keV}$, which is wide considering that the corresponding mean lifetime is $T = 4.4 \times 10^{-21}$ s. This state has such a very short lifetime that physically it may be considered a part of the continuum background. This explains why the real part of the Gamov function, which we call $G(r)$ in figure 3, is extended rather far from the radius of the nucleus, which in this case is 7.1 fm. Yet, the imaginary part of this function is very small in the spatial region covered by the figure.

We evaluated the corresponding function according to our formalism searching for the residues of the scattering function $\omega^{+}(r)$ making use of the computer code of [28]. As seen in figure 3, these functions coincide with each other within the precision of the figure. This is expected, as shown in [29].

7. Conclusions

Gamov resonances have been introduced in modern physics at the very beginning of quantum mechanics [1]. Yet, their inclusion in the theory have been hindered by many difficulties, particularly by the fact that the Gamov functions diverge large distances and cannot be normalized within the Hilbert space.

We have presented in this paper a mathematical structure to define the Gamov states which do not have any of the shortcomings mentioned above. We have thus shown that the analysis of processes taking place in the continuum part of the quantum spectra can conveniently be performed by introducing Gamov states. This was achieved by describing the Gamov states as *generalized* states. However, isolated Gamov vectors are not by themselves physical objects since we have shown that their 'traces' and energies both vanish within quantum mechanics. Using other formalisms [7, 8, 11, 17] one succeeds in defining a norm [6, 9–11] but in a space which is not the Hilbert space and thus outside quantum mechanics. These formalisms have shown to be very fruitful for understanding the structure of resonances and, in general, processes occurring in the continuum part of the quantum spectra, although one obtains unphysical properties as complex probabilities [16, 17]. In the formalism presented here the Gamov vectors are useful tools to construct spectral decompositions of the time evolution. In fact these kind of mathematical objects are not an exception in scattering theory, where neither plane waves are physical objects since their norms are not finite. Plane waves, as Gamov vectors, do not belong to the Hilbert space, but are linear functionals on convenient Hilbert subspaces.

For the real spectral decomposition of our formalism, the generalized states $(\widetilde{\Phi}_{\omega\omega'}|$ of the real spectral decomposition satisfy $(\widetilde{\Phi}_{\omega\omega'}|I) = 0$ and $(\widetilde{\Phi}_{\omega\omega'}|H) = 0$, i.e. they have zero values of the energy and the generalized trace. However, the $(\widetilde{\Phi}_{\omega\omega'}|$ are very important, because they provide the time-dependent part of any physical state functional. The energy and the non-zero part of the generalized trace is carried out by the time-independent components $(\widetilde{\Phi}_{\omega}|$, satisfying $(\widetilde{\Phi}_{\omega}|H) = \omega$ and $(\widetilde{\Phi}_{\omega}|I) = 1$. Therefore both $(\widetilde{\Phi}_{\omega}|$ and $(\widetilde{\Phi}_{\omega\omega'}|$ components are physically relevant parts of a state functional.

For the complex spectral decomposition, the generalized states $(\widetilde{\Phi}_{00}|$, $(\widetilde{\Phi}_{0z'}|$, $(\widetilde{\Phi}_{z0}|$ and $(\widetilde{\Phi}_{zz'}|$, obtained by analytic extensions of the functionals $(\widetilde{\Phi}_{\omega\omega'}|$, have also zero energy and zero 'trace'. They expand the time-dependent part of the state functionals.

Within our formalism Gamov states play a fundamental role in describing structures in the continuum spectrum, i.e. resonances which appear as a natural extension of bound states. We have shown that if a resonance is narrow enough then the imaginary part of the Gamov energy is, as usual, related to the width of the resonance. But we have also shown that if the resonance is wide the Gamov vector may still play an important role, although that relation wanes. In our formalism probabilities are always real quantities which have standard quantum mechanical meaning. Therefore we are not confronted with the problem of avoiding the inclusion of wide resonances in the formalism, which otherwise induces large complex probabilities. Due to the quantum mechanics framework on which our formalism is built, all resonances are treated on the same footing.

We thus think that we have solved the age-old problem of including Gamov states properly, from the point of view of the scattering theory, to treat quantum processes in the continuum.

Appendix A. Adjoint functionals

(i) Proof of the relation $\langle z| = \langle\widetilde{\overline{z}}|$.

Starting from the definition $\langle z|\varphi\rangle \equiv \overline{\langle\varphi|z\rangle}$ given in equation (20) and using equations (17) and (19), we obtain

$$\langle z|\varphi\rangle = \overline{\langle\varphi|z\rangle} = \overline{\overline{\varphi(\overline{z})}} = \varphi(\overline{z}) = \langle\widetilde{\overline{z}}|\varphi\rangle. \tag{A1}$$

If these equalities hold for arbitrary 'test vectors' φ, we deduce

$$\langle z| = \langle\widetilde{\overline{z}}|. \tag{A2}$$

(ii) Proof of the relation $|\widetilde{z}\rangle = |\overline{z}\rangle$.

Starting from the definition $\langle\varphi|\widetilde{z}\rangle \equiv \overline{\langle\widetilde{z}|\varphi\rangle}$ given in equation (20) and using equations (17) and (19), we obtain

$$\langle\varphi|\widetilde{z}\rangle = \overline{\langle\widetilde{z}|\varphi\rangle} = \overline{\varphi(z)} = \overline{\varphi(\overline{\overline{z}})} = \langle\varphi|\overline{z}\rangle. \tag{A3}$$

Therefore, for arbitrary 'test vectors' φ we deduce

$$|\widetilde{z}\rangle = |\overline{z}\rangle. \tag{A4}$$

(iii) Proof of the relation $\langle z^{+}| = \langle\widetilde{\overline{z}}^{+}|$.

Starting from the definition $\langle z^{+}|\varphi\rangle \equiv \overline{\langle\varphi|z^{+}\rangle}$, and using equations (25), (B3) and (A2), we obtain

$$\begin{aligned}\langle z^{+}|\varphi\rangle &= \overline{\langle\varphi|z^{+}\rangle} = \overline{\langle\varphi|z\rangle} + \overline{\langle\varphi|R^{+}(z)V|z\rangle} = \langle z|\varphi\rangle + \langle z|VR^{-}(\overline{z})|\varphi\rangle \\ &= \left[\langle\widetilde{\overline{z}}| + \langle\widetilde{\overline{z}}|VR^{-}(\overline{z})\right]|\varphi\rangle = \langle\widetilde{\overline{z}}^{+}|\varphi\rangle.\end{aligned} \tag{A5}$$

Therefore we obtain

$$\langle z^{+}| = \langle\widetilde{\overline{z}}^{+}|. \tag{A6}$$

(iv) Proof of the relation $|\widetilde{z}^{+}\rangle = |\overline{z}^{+}\rangle$.

Starting from the definition $\langle\varphi|\widetilde{z}^{+}\rangle \equiv \overline{\langle\widetilde{z}^{+}|\varphi\rangle}$ and using equations (25), (B3) and (A4), we obtain

$$\begin{aligned}\langle\varphi|\widetilde{z}^{+}\rangle &= \overline{\langle\widetilde{z}^{+}|\varphi\rangle} = \overline{\langle\widetilde{z}|\varphi\rangle} + \overline{\langle\widetilde{z}|VR^{-}(z)|\varphi\rangle} = \langle\varphi|\widetilde{z}\rangle + \langle\varphi|R^{+}(\overline{z})V|\widetilde{z}\rangle \\ &= \langle\varphi|[|\overline{z}\rangle + R^{+}(\overline{z})V|\overline{z}\rangle] = \langle\varphi|\overline{z}^{+}\rangle\end{aligned} \tag{A7}$$

and therefore

$$|\widetilde{z}^{+}\rangle = |\overline{z}^{+}\rangle. \tag{A8}$$

Appendix B. Properties of the analytic extensions $R^{+}(z)$ and $R^{-}(z)$ of the resolvent

In this appendix we will prove that if $R^{+}(z)$ has a pole at point z_0, then $R^{-}(z)$ has a pole at point $\overline{z}_0$.

Let us first consider $z \in \mathbb{C}^{-}$ (the lower half of the complex plane), and two test functions φ and ψ. Then we have

$$\begin{aligned}\langle\psi|R^{-}(\overline{z})|\varphi\rangle &= \mathrm{cont}_{s\in\mathbb{C}^{-}\to\overline{z}}\langle\psi|(s-H)^{-1}|\varphi\rangle \\ &= \mathrm{cont}_{s\in\mathbb{C}^{-}\to\overline{z}}\langle(\overline{s}-H)^{-1}\psi|\varphi\rangle \\ &= \mathrm{cont}_{s\in\mathbb{C}^{-}\to\overline{z}}\overline{\langle\varphi|(\overline{s}-H)^{-1}\psi\rangle} \\ &= \overline{\mathrm{cont}_{\overline{s}\in\mathbb{C}^{+}\to z}\langle\varphi|(\overline{s}-H)^{-1}\psi\rangle} = \overline{\langle\varphi|R^{+}(z)|\psi\rangle}.\end{aligned} \tag{B1}$$

If $z \in \mathbb{C}^{+}$, we obtain

$$\begin{aligned}\langle\psi|R^{-}(\overline{z})|\varphi\rangle &= \langle\psi|R(\overline{z})|\varphi\rangle = \langle R(\overline{z})^{\dagger}\psi|\varphi\rangle \\ &= \langle R(z)\psi|\varphi\rangle = \overline{\langle\varphi|R(z)\psi\rangle} = \overline{\langle\varphi|R^{+}(z)|\psi\rangle}.\end{aligned} \tag{B2}$$

From equations (B1) and (B2), we deduce the relation

$$\langle\psi|R^{-}(\overline{z})|\varphi\rangle = \overline{\langle\varphi|R^{+}(z)|\psi\rangle} \qquad \text{for all} \quad z. \tag{B3}$$

It is clear from this relation that if $R^{+}(z)$ has a pole at z_0, then $R^{-}(z)$ has a pole at $\overline{z}_0$.

Appendix C. Complex eigenvalues of the Hamiltonian

(i) Proof of the eigenvalue equation $\langle \widetilde{f}_0|H = z_0\langle \widetilde{f}_0|$.

Starting with the definition $\langle \widetilde{f}_0|\varphi\rangle \equiv \mathrm{cont}_{\omega'\to z_0}\langle \omega'^{+}|\varphi\rangle$ of the vector $\langle \widetilde{f}_0|$, we can replace φ by $H\varphi$ to obtain

$$\begin{aligned}\langle \widetilde{f}_0|H\varphi\rangle &= \mathrm{cont}_{\omega'\to z_0}\langle \omega'^{+}|H\varphi\rangle = \mathrm{cont}_{\omega'\to z_0}\omega'\langle \omega'^{+}|\varphi\rangle \\ &= z_0\,\mathrm{cont}_{\omega'\to z_0}\langle \omega'^{+}|\varphi\rangle = z_0\langle \widetilde{f}_0|\varphi\rangle.\end{aligned} \tag{C1}$$

If these equalities hold for arbitrary 'test vectors' φ, we deduce

$$\langle \widetilde{f}_0|H = z_0\langle \widetilde{f}_0|. \tag{C2}$$

(ii) Proof of the eigenvalue equation $H|f_0\rangle = z_0|f_0\rangle$.

Starting with the definition $\langle \psi|f_0\rangle \equiv (-2\pi\mathrm{i})\,\mathrm{cont}_{\omega'\to z_0}(\omega' - z_0)\langle \psi|\omega'^{+}\rangle$ of the vector $|f_0\rangle$, we can replace ψ by $H\psi$ to obtain

$$\begin{aligned}\langle H\psi|f_0\rangle &\equiv (-2\pi\mathrm{i})\,\mathrm{cont}_{\omega'\to z_0}(\omega' - z_0)\langle H\psi|\omega'^{+}\rangle \\ &= (-2\pi\mathrm{i})\,\mathrm{cont}_{\omega'\to z_0}(\omega' - z_0)\omega'\langle \psi|\omega'^{+}\rangle \\ &= z_0(-2\pi\mathrm{i})\,\mathrm{cont}_{\omega'\to z_0}(\omega' - z_0)\langle \psi|\omega'^{+}\rangle = z_0\langle \psi|f_0\rangle.\end{aligned} \tag{C3}$$

By definition, the action of H on the generalized vector $|f_0\rangle$ is defined through the relation $\langle \psi|Hf_0\rangle \equiv \langle H^{\dagger}\psi|f_0\rangle$. As the Hamiltonian is self-adjoint we have also $H^{\dagger}\psi = H\psi$. Therefore equation (C3) gives

$$H|f_0\rangle = z_0|f_0\rangle. \tag{C4}$$

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